Properly Extending Properly n-REA Sets

Peter M. Gerdes

New England Recursion and Definability Seminar 2020

Outline

1. Background
2. Properly Extending 2-REA Sets
3. Non-Extendable 3-REA Set

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2 Properly Extending 2-REA Sets

3 Non-Extendable 3-REA Set

REA sets

- $A^{[n]}$ is the n-th column of A and $A^{[\leq n]}$ is the restriction of A to the first n columns.
- The i-th hop is $\mathcal{H}_i(A) \stackrel{\text{def}}{=} A \oplus W_i^A$. REAin A is a synonym for is a hop of A.
- $\emptyset$ is 0-REA and if A is n-REA then $\mathcal{H}_i(A)$ is $n+1$-REA.
- A set is properly $(n+1)$-REA just if it is $n+1$-REA and not Turing equivalent to any n-REA set.

- We identify n-REA sets with n-column sets where the $l+1$-st column is r.e. in the first l columns.
- We will denote the n-REA set with index e by X_e.

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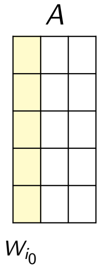

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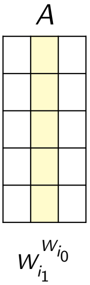

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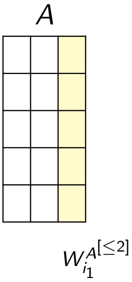

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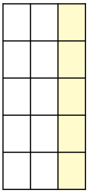

Axioms

- Handwaving details consider an approximation to a 3-REA set A.
- 1 enumerated into 3-rd column dependent on highlighted area.
- Enumeration of 1 **cancels** 1
- 1 cancels 1 restoring 1
- Can effectively identify n-REA sets with r.e. sets of axioms (enumerate y into $A^{[n]}$ if $\sigma \prec A^{[<n]}$).

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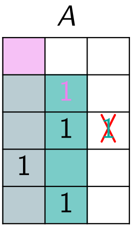

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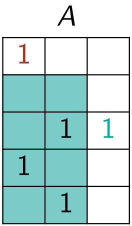

Proper Extendability

Question

Can every properly n-REA set A be extended to a properly $n+1$-REA set $\mathcal{H}_i(A)$?

Prior Results

- Trivially true for $n = 0$
- The claim is true for $n = 1$ (Soare and Stob 1982)
- The claim is true for $n = 2$ (Cholak and Hinman 1994).

Novel Result with Peter Cholak

Claim fails at $n = 3$.

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2-REA Proper Extendability

Proposition (Cholak and Hinman 1994)

Every properly 2-REA can be extended to a properly 3-REA set.

Build A r.e. in proper 2-REA C meeting (where X_e is 2-REA):

Requirements

$\mathscr{Q}_{j,e}$: $\left(\phi_j^{C \oplus A} \neq X_e \vee \phi_j^{X_e} \neq C \oplus A\right)$

- We think of $C \oplus A$ as a 3 column set.
- Can find j so ϕ_j^Z switches computation based on $Z = X_e$ or $Z = C \oplus A$.

Let's start easy and suppose we control C. How would we build $Z = C \oplus A$ to be properly 3-REA set.

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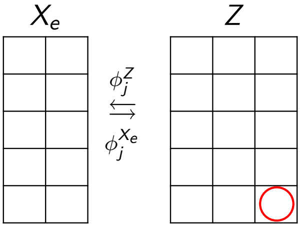
Building Properly 3-REA

Meet one requirement for Z: $\phi_j^Z \neq X_e \vee \phi_j^{X_e} \neq Z$



- Hold $\textcircled{z_3}$ out of Z (red for disagree).
- Await agreement. **Gray X_e area use closed**.
- Put $\textcircled{z_3}$ in Z. Await agreement.
- Some x_2 must enter X_e.
- Extend agreement. x_2 use included for use closure.
- Cancel $\textcircled{z_3}$ by enumerating z_2.
- Restores computation with $X_e(x_2) = 0$. Await Agreement.
- Some x_1 must cancel x_2 to agree.
- Cancel z_2 with z_1. Restoring comp: $X_e(x_1) = 0$. **Permanent Disagreement**.

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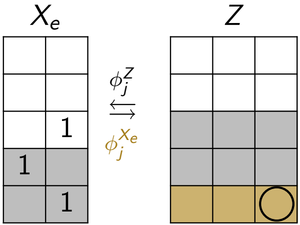

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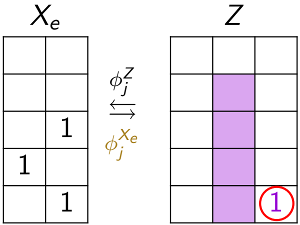

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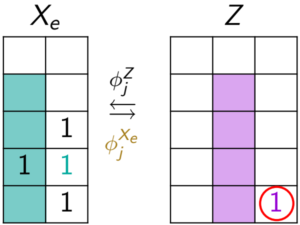

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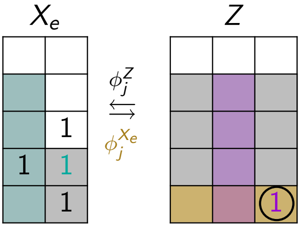

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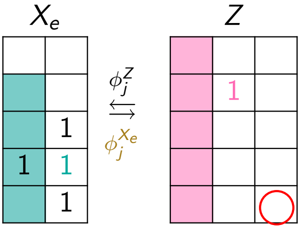

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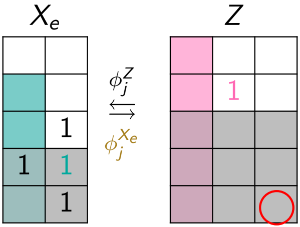

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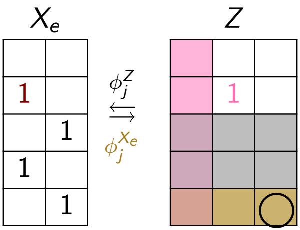

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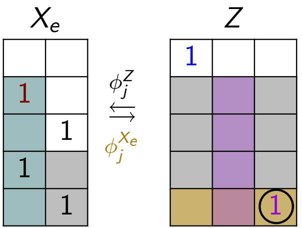

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Extending Properly 2-REA

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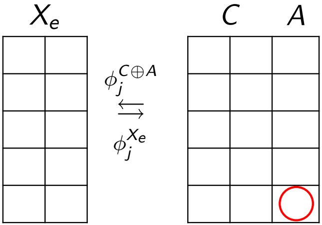

- Problem: C might not supply z_2.
- **Assume:** build $z_3^n, n \in \omega$ so all late ($C^{[1]}$ comp modulus) enums into $C^{[2]}$ work as some z_2^n.
- WIN If X_e doesn't cancel (in r.e. proof couldn't)

- Undoing z_2^n enum (restoring prior agreement) gives **WIN**.
- Otherwise $C^{[1]} \oplus X_e{}^{[1]}$ recovers C since $C^{[2]}$ enum ensures $X_e{}^{[1]}$ change **WIN**

$\leq_T$: $\mathscr{Q}_{j,e}$ acts infinitely so $C \equiv_T C \oplus A \equiv_T X_e \geq_T X_e{}^{[1]}$

$\geq_T$: Every late (not before $C^{[1]}$ modulus) entry into $C^{[2]}$ serves as some z_2^n causing change to $X_e{}^{[1]}$ below bound set when z_3^n enumerated.

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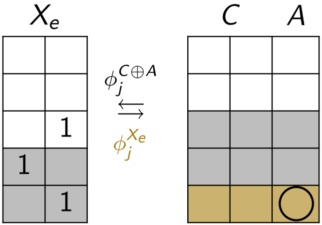

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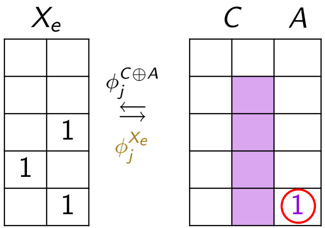

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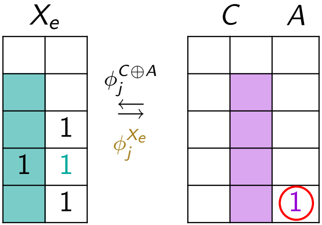

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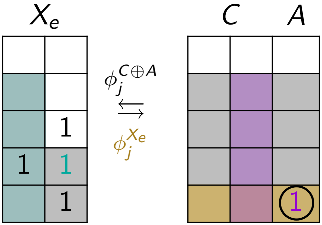

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Extending Properly 2-REA

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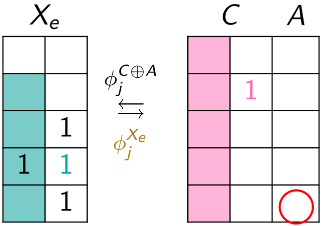

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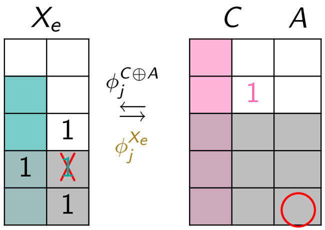

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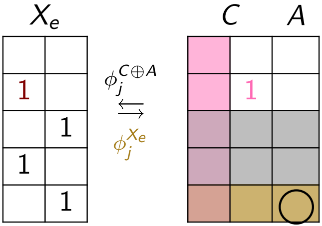

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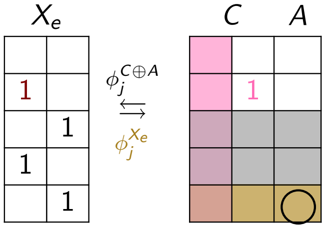

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No Uniform Proper Extendability

If z_3^n choice (Assume) existed result would be uniform. **It's not!**

Proposition (Cholak and Hinman 1994)

For all $n > 0$, total computable p there is a properly n-REA set X_e such that $\mathcal{H}_{p(e)}(X_e)$ is not properly n-REA

Proof.

- Build $X_e = \mathcal{H}_e\left(\mathbb{0}^{(n-1)}\right)$ to frustrate p. Assume we know $j = p(e)$.
- Let h (Hop inversion Jockusch and Shore 1983) satisfy $\mathcal{H}_j\left(X_{h(j)}\right) \equiv_T \mathbb{0}^{(n)}$.
- By fixed point let j s.t. $W_j^Z = W_{p(h(j))}^Z$ and $e = h(j)$.
- Hence $\mathcal{H}_{p(e)}(X_e) = \mathcal{H}_{p(h(j))}\left(X_{h(j)}\right) = \mathcal{H}_j\left(X_{h(j)}\right) \equiv_T \mathbb{0}^{(n)}$

□

Non-uniform Approach

Idea

Build A_0, A_1 so that one of $C \oplus A_i$ is properly 3-REA.

Requirements

$\mathscr{Q}_{e_0,e_1,j}$: $(\exists k)\left(\phi_j^{C \oplus A_k} \neq X_{e_k} \vee \phi_j^{X_{e_k}} \neq C \oplus A_k\right)$

Idea

Chose $z_3^{n,k}$ for A_k and interleave so that:

1. *Sequence infinite iff $\neg\mathscr{Q}_{e_0,e_1,j}$.* (Only stop on disagree)
2. *Any late enum into $C^{[2]}$ acts as $z_2^{m,k'}$, i.e., cancels $z_3^{m,k'}$.*
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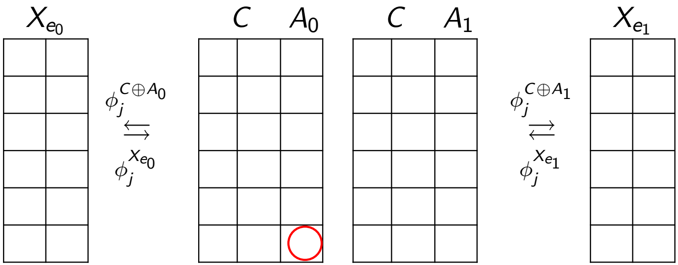

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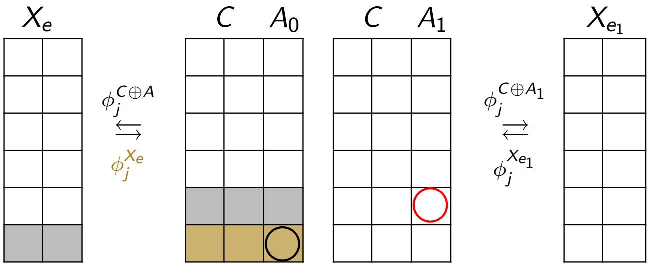

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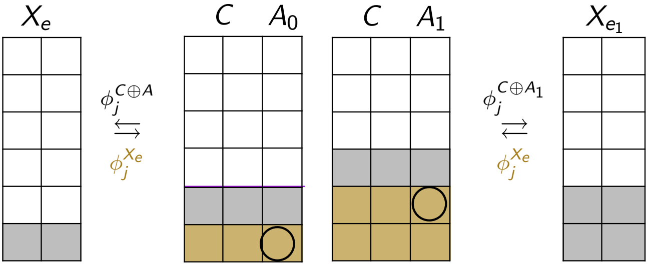

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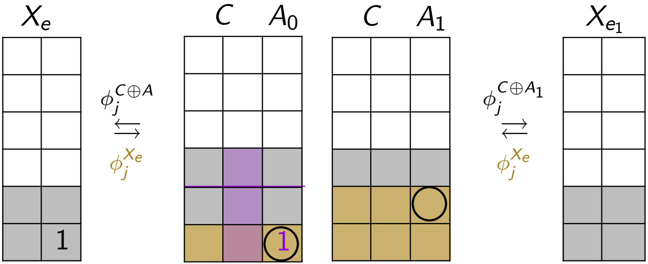

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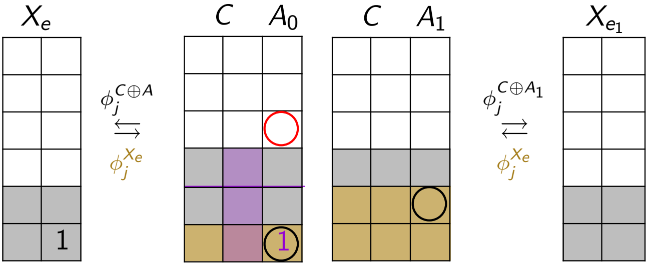

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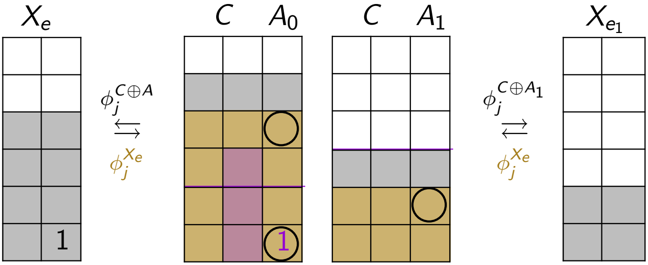

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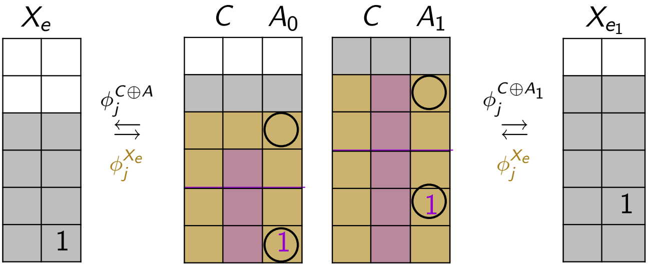

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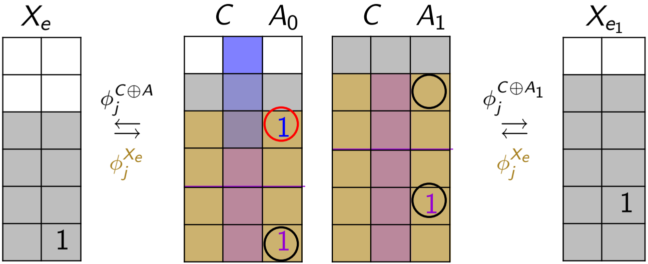

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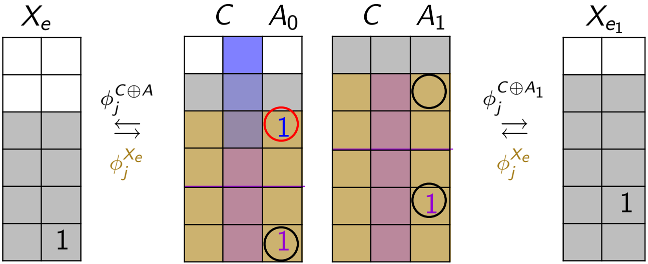

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Outline

1 Background

2 Properly Extending 2-REA Sets

3 Non-Extendable 3-REA Set

Novel Result

Theorem (Novel Result with Peter Cholak)

There is a properly 3-REA set A which can't be extended to a properly 4-REA set $\mathcal{H}_i(A)$.

Build A, Y_i 3-REA Γ_i, Θ to satisfy: (where X_e is 2-REA)

Requirements

$\mathscr{P}_i$: $\Gamma_i(\mathcal{H}_i(A)) = Y_i \wedge \Theta(Y_i) = W_i^A$

$\mathscr{R}_{j,e}$: $\Phi_j(A) \neq X_e \vee \Phi_j(X_e) \neq A$

- $\mathscr{P}_i$ ensures that $A \overline{\oplus} Y_i \stackrel{\text{def}}{=} \bigoplus_{k \leq 3} A^{[k]} \oplus Y_i^{[k]}$ is 3-REA set equivalent to $\mathcal{H}_i(A)$
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- Use finite injury method to build A, Y_i as limit of approximations.
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- Enemy (W_i^A) wants to walk changes 'up' columns of Y_i till can't match
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Try (in order) many options c_n for c. We have option to cancel c_k and 'time travel' to point in time right before enumerating c_k. Enemy will run out of different ways to enumerate into $W_i^A, i < \langle\langle j, e \rangle\rangle$.

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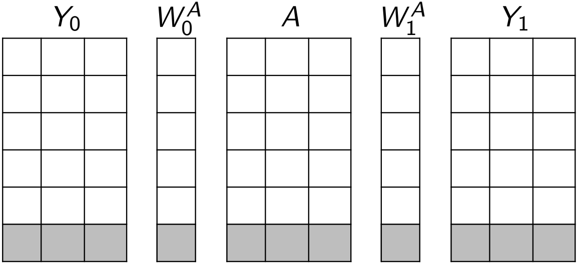

Functionals defined on some initial use.

- At s_{-1} $\mathcal{R}_{j,e}$ chooses c_0 large in $A^{[3]}$.
- At s_0 c_0 enters $A^{[3]}$ resetting W_i^A
- At s_1 c_1 enters again resetting W_i^A to s_{-1} state.
- We cancel 1 by enumerating 1 AGREEMENT
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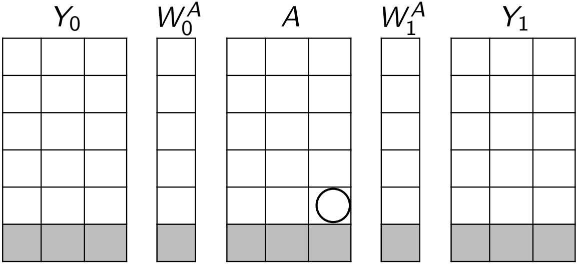

Ignore all elements but one (call q) entering/leaving W_i^A for now.

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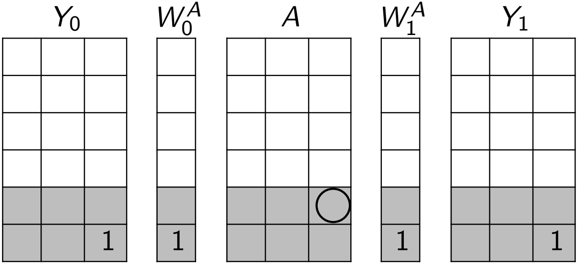

Elements enter W_i^A while waiting to see agreement with X_e.

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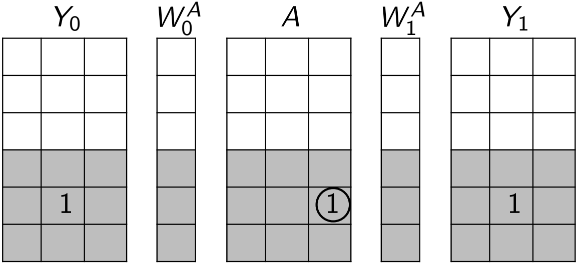

Must cancel enumerations into Y_i to agree with prior computation.

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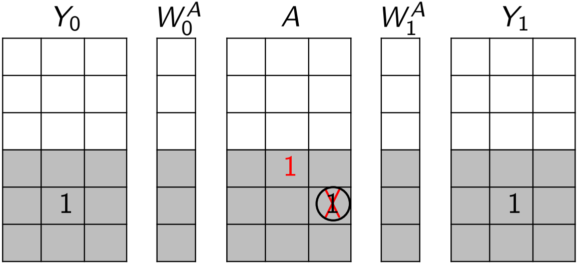

Interlude: What if we tried to use c_0 as c by enum b_0 into $A^{[2]}$

- At s_{-1} $\mathcal{R}_{j,e}$ chooses $ⓒ_0$ large in $A^{[3]}$.
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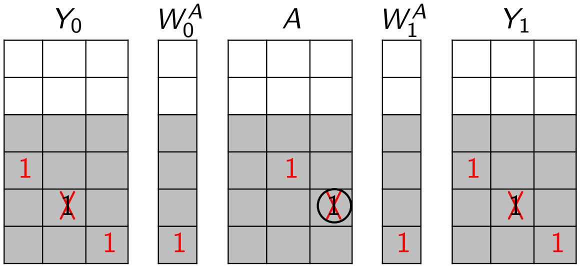

Interlude: q returned to W_i^A, Y_i. Cancelling b_0 would break functionals.

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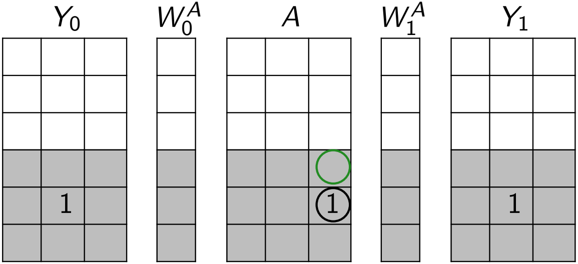

Instead we wait for X_e agree through c_1

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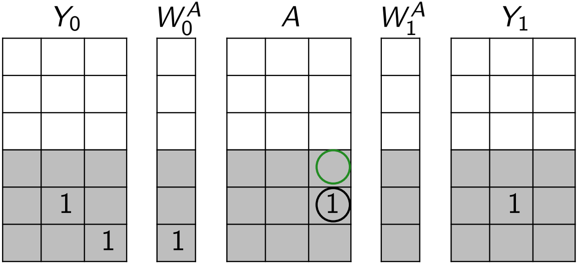
- At s_1 c_1 enters again resetting W_i^A to s_{-1} state.
- We cancel 1 by enumerating 1 AGREEMENT
- Enum b_1 canceling c_1 but not $Y_i^{[\leq 2]}$. Enum a_1 VICTORY

Basic $\mathcal{R}_{j,e}$ Action



During wait q enters W_0^A changing Y_0 .

- At s_{-1} $\mathcal{R}_{j,e}$ chooses c_0 large in $A^{[3]}$.
- At s_0 c_0 enters $A^{[3]}$ resetting W_i^A
- At s_1 c_1 enters again resetting W_i^A to s_{-1} state.
- We cancel 1 by enumerating 1 AGREEMENT
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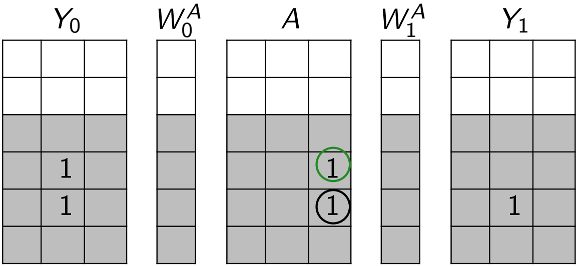
Basic $\mathcal{R}_{j,e}$ Action



c_1 cancels q from W_0^A changing Y_0 not Y_1

- At s_{-1} $\mathcal{R}_{j,e}$ chooses c_0 large in $A^{[3]}$.
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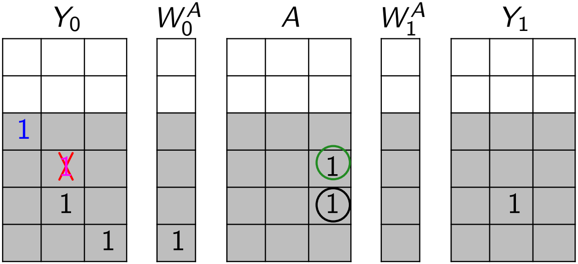

q enum into W_0^A. Restore old state of Y_0 don't re-enum code for q.

- At s_{-1} $\mathcal{R}_{j,e}$ chooses c_0 large in $A^{[3]}$.
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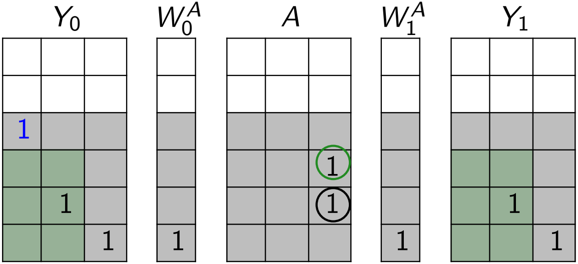
- We cancel 1 by enumerating 1 **AGREEMENT**
- Enum b_1 canceling c_1 but not $Y_i^{[\leq 2]}$. Enum a_1 **VICTORY**

Basic $\mathcal{R}_{j,e}$ Action



Only enum into $Y_i^{[3]}$ untill $\mathscr{R}_{j,e}$ expansionary.

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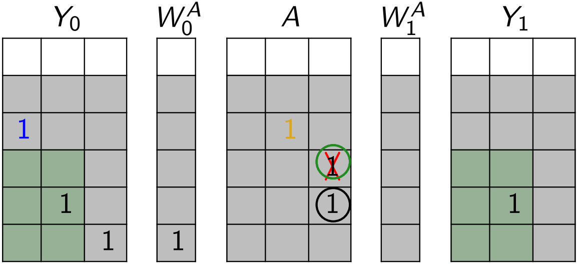
Basic $\mathcal{R}_{j,e}$ Action



Automatically roll back $Y_i^{[3]}$ since $A\overline{\oplus}Y_i$ 3-REA.

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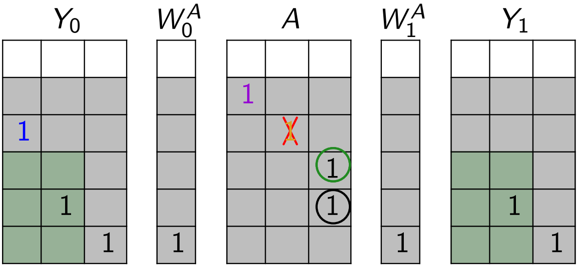
Basic $\mathcal{R}_{j,e}$ Action



Wait for $\mathscr{R}_{j,e}$ expansionary (again only modify $Y_i^{[3]}$) before flipflop.

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- At s_0 c_0 enters $A^{[3]}$ resetting W_i^A
- At s_1 c_1 enters again resetting W_i^A to s_{-1} state.
- We cancel 1 by enumerating 1 **AGREEMENT**
- Enum b_1 canceling c_1 but not $Y_i^{[\leq 2]}$. Enum a_1 **VICTORY**

It's never *that* simple

- When we see q enter W_i^A we have to choose between keeping our options open or jumping back to agree with a long past stage $s_k - 1$ at the cost giving up change to agree with intervening $s_{k'} - 1, k' > k$
- We must decide how to respond with Y_i immediately after enumeration. Enemy can decide what set $W_{i'}^A$ to enumerate into next based on our choices so far.

Turns out clever enemy can beat most obvious ways to try and ensure agreement with the past.

- We give a second priority argument to bound number of $\mathscr{R}_{j,e}$ expansionary stages before victory.
- Turns out that considering more than one element (e.g. all $x < s_{-1}$) isn't much different than considering more sets W_i^A with Γ_i of higher priority.

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Final Notes

- Lots of open questions regarding REAsets. Come and play!
    - Still lots of easy to state open questions (I'm kinda obsessed with existence of minimal ω-REA arithmetic degree but I keep running into nice problems for small n n-REA sets)
- My rec-thy package for $\LaTeX 2_\varepsilon$ is at an early beta stage and feedback is welcome.

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